CHILDREN'S UNDERSTANDING OF THE NUMBER SYSTEM

Noel Thomas
Charles Sturt University Bathurst
<nthomas@csu.edu.au>

Joanne Mulligan
Macquarie University Sydney
<jmull@ted.educ.mq.au>

A cross-sectional study of 132 NSW rural children from grades K-6 assessed counting, number sense, grouping/partitioning, regrouping, place value and structure of the number system. Task-based interview data exhibited lack of understanding of the base ten system, with little progress made during Grades 5 and 6. Few Grade 6 children used holistic strategies or generalised the structure of the number system. Grouping strategies were not well linked to formation of multiunits; additive rather than multiplicative relations dominated the interpretation of multidigit numbers.

Understanding the multiplicative nature of the base 10 system is critical to the development of numeration, place value and number sense. A critical problem is that children do not recognise that the numbers they use are part of a system, and thus they do not have the multiunit structures to understand how the numbers are regrouped in mental and written algorithms. Further, understanding of the use of powers of ten is needed in order to construct the multiunit conceptual structures for multidigit numbers.

RESEARCH ON NUMERATION AND PLACE VALUE

There has been continuing strong research interest in children's development of numeration and place value. Several studies have described the development of 'ten as a unit' (Boulton-Lewis & Halford, 1992; Cobb & Wheatley, 1988; Steffe & Cobb, 1988; Thompson, 1982), and the understanding of place value (Bednarz & Janvier, 1988; Kamii, M., 1982; Ross, 1989, 1990; Thompson, 1992). It is suggested that the construction of new conceptual multiunit structures is an ongoing process that occurs within the classroom environment that includes many elements other than just the representational objects for number (Fuson, Wearne, Hiebert, Murray, Human, Olivier, Carpenter, & Fennema 1997).

The importance of mental models that reflect conceptual structure of numeration has been highlighted by Boulton-Lewis and Halford (1992). From an information processing model, Boulton-Lewis and Halford (1992) pursued the question of how children were influenced by concrete analogs in their developing numeration and place value skills, particularly in relation to counting and place value. At the same time, Hiebert and Wearne (1992) investigated children's representations of numeration including their understanding of decimals. Other studies endeavoured to categorise in developmental levels, key aspects of place value knowledge (Jones, Thornton & Putt, 1994; Resnick, 1983; Ross, 1990). However, few studies have consistently examined the relative influence of these key elements of children's understanding of the structure of number system.

Classroom-based Studies

Much of the groundwork focusing on children's difficulties with numeration and place value has given rise to classroom-based studies (Bednarz & Janvier, 1988; Carpenter, Fennema & Romberg, 1993; Fuson, Fraivig & Burghardt, 1992; Hiebert & Wearne, 1992) and learning frameworks (Denvir & Brown, 1986a, b; Fuson et al., 1997; Jones, Thornton, Putt, Hill, Mogill, Rich & Van Zoest, 1996; Resnick, 1983) aimed at improving the teaching and learning of numeration, place value and multidigit operations. Case studies and teaching experiments have investigated young children's development of two and three-digit numeration (Cobb & Bauersfeld, 1995; Jones et al., 1994).

New Directions for Research on Numeration and Place Value

From the developing body of research on numeration and place value we know that a child's understanding of the numeration system is complex, is not necessarily lock-step, and develops over many years. There is also a fundamental change in the way a child understands number from the early notion of number as a counting unit, to the construction of composite units (Steffe, 1994) and the reinitialising of units (Confrey, 1994). The process that starts with treating a collection as a whole and then develops as a system that is built on the iteration of grouping collections, requires significant cognitive reorientations.

Research on how children extend their early number understanding and skills to cater for the expanding system of generating number names and symbols that we know as the Hindu-Arabic numeration system has been far less comprehensive than the work on early number. In the early 1990's there seemed a need to understand more fully why children failed to develop a structure for the numeration system at a time when concrete materials and more conceptually-based teaching approaches had been advocated for over a decade in Australia (Booker, Irons & Jones, 1980; Dienes, 1960; 1964).

Despite much research on counting and place value in the 1980's, by the 1990's researchers could not assert any firm explanations about why children fail to grasp the structure of the number system. Sinclair, Garin, & Tieche-Christinat (1992) make the crucial point that:

> Understanding place value is not a matter of simply 'cracking' an arbitrary written code following adult explanation or some degree of exposure to computation. It is indissolubly linked to understanding the number system itself. Grasping it implies understanding a multiplicative recursive structure. (p. 93)

This study investigated which aspects of developing number knowledge contribute to the apparent failure of children to make sense of numeration as a number system. Two broad research questions were addressed. What strategies do children use in solving numeration tasks involving the elements of counting, grouping, and structuring place value? How are critical aspects of counting and grouping related to understanding the base ten structure of the numeration system?

These questions are critical because children need structural flexibility in counting and grouping in order to operate meaningfully with the number system. The role of visualisation of the counting sequence was also examined in view of children's representations of the numeration system which has been reported elsewhere (Thomas & Mulligan, 1995; Thomas, Mulligan & Goldin; 1996)

METHOD

The study was designed as a broad exploratory investigation employing task-based interviews and quantitative and qualitative methods of analysis. A descriptive approach was used to provide evidence of qualitative differences in the way children use their strategies and relate key elements of the numeration system.

Sample: A cross-sectional sample of 132 children from Grades K to 6 was randomly selected from six Government schools in the Western Region of New South Wales. Five of the schools were from three large regional towns and one was the only school in a small rural town. The sample was representative of a wide range of socio-economic backgrounds.

Interview tasks: A total of eighty-nine tasks were incorporated after trialling in a pilot study. The tasks were designed to probe understanding of numeration through: counting; grouping/ partitioning; regrouping, place value; structure of numeration and number sense. Many of the tasks were refined from those used by previous researchers (Bednarz & Janvier, 1988; Cobb & Wheatley, 1988; Davydov, 1982; Denvir & Brown, 1986; Labinowicz, 1985; Mulligan, 1992; Ross, 1986; Steffe & Cobb, 1988; Wright, 1991). The tasks were graded by level of difficulty and different subsets of tasks were given to each grade cohort.

Interview Procedures: All interviews were conducted by the researcher in October and November of the school year and these were carried out in a small room separate from the classroom at the schools. The interviewer told the children that he was interested in how they worked out the answers to some questions. The tasks were presented verbally to the children, with concrete or pictorial material. The tasks were re-read to the child as often as necessary to assist him/her in understanding the intention of the question. Concrete materials and pencil and paper were available on the table. The interviewer explained to the child that this material was available for use if required and that he/she was to 'think aloud' as the 'maths activities' were being done. The child wase praised for their attempts, but no feedback was given as to the correctness of their responses. When a response was unclear, follow-up neutral questions were asked by the interviewer such as: "Can you tell me how you did that?"; "Can you describe what you did there?" and "Did you see anything in your mind when you did that?". The interviews were audio-taped and the length of time for each interview ranged from 25 to 65 minutes.

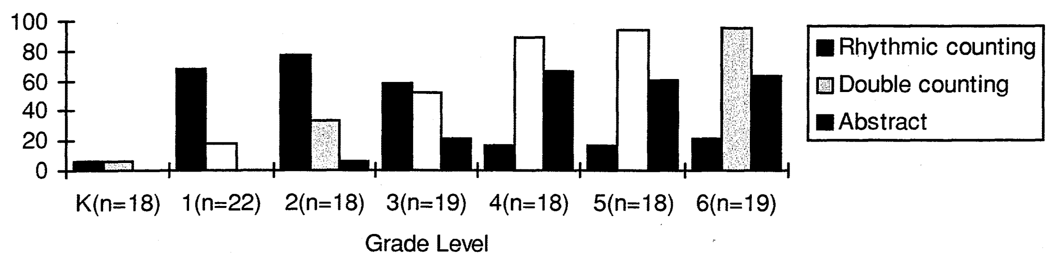
Analysis of Data: Item Response Analysis using Student-Problem curve theory (Harnish, 1983) and the Rasch model (Rasch, 1980) were used initially to obtain some overall measure of student performance in Grades 4 to 6. The main analysis of results involved coding responses for student performance and strategy use (which is reported partially in this paper) across tasks and grades. The coding of responses was trialled in a pilot study which was devised to indicate correct, incorrect or non-response to the tasks. Strategies used for both incorrect and correct responses to tasks were coded in order to classify the range of numeration skills and understandings. Re-coding was conducted by two independent coders for 20% of responses which established a high level of intercoder reliability (0.92).

ANALYSIS OF RESULTS

The discussion will provide an overview of key findings based on children's performance and an analysis of their strategies. The majority of children across Grades 1-6 recognised and used concrete materials to represent grouping of numbers, could identify place values of digits in numerals, and could successfully carry out algorithmic procedures. However, many relied on unitary counting in mental calculations and they did not necessarily use structured materials meaningfully. Children may have shown good performance on 2-digit mental calculations, but generally the use of unitary counting methods prevailed and so many children could not extend their successful use of small numbers to larger numbers. There was, in general, a weak awareness of structure and, in particular, of the multiplicative nature of this structure. It appears from the data that additive relationships within the number system are better understood and used than multiplicative relationships. The lack of conceptual understanding of the tens and hundreds structure of number means that the knowledge of ones, tens and hundreds that exists is not connected and so ability to work with larger numbers is restricted. On the other hand, some young children acquired elements of understanding of place value, represented number in ways that reflected elements of structure and developed their own efficient mental strategies.

Children's counting abilities were shown to be of fundamental importance to developing understanding of the number system. Figure 1 shows that by the end of Grade 2, most children still had a strong reliance on rhythmic counting but other children had developed double counting skills. For Grade 5 and 6 children there was little discernible progress with rhythmic or double counting.

Figure 1 *Counting Task 5: Counting by Threes*
Solution Strategies for Counting Task 5, Counting in Threes: Percentage of Sample Giving Correct Responses, by Strategy use.



At Grade 2, Figure 2 shows the majority of children (67%) were undertaking mental calculations using unitary counting methods for tasks involving 1-digit numbers. At Grade 4 there were still 11% of children using counting-on by ones.

Figure 2 Regrouping Task :1 Mental addition (43+8) represented by Pregrouped Material Solution Strategies for Regrouping Task 1, Addition: Percentage of Sample Giving Correct Responses, by Strategy use.

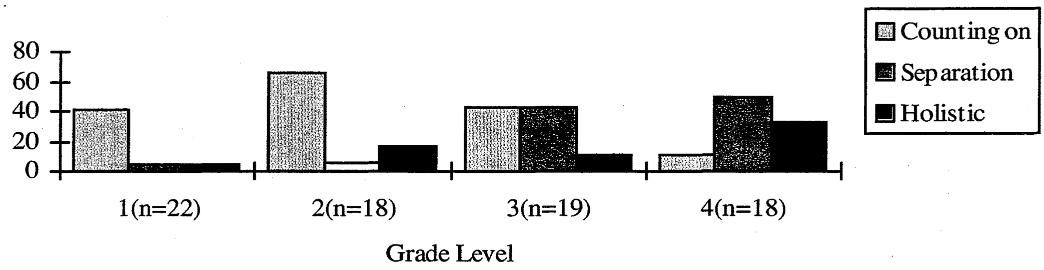
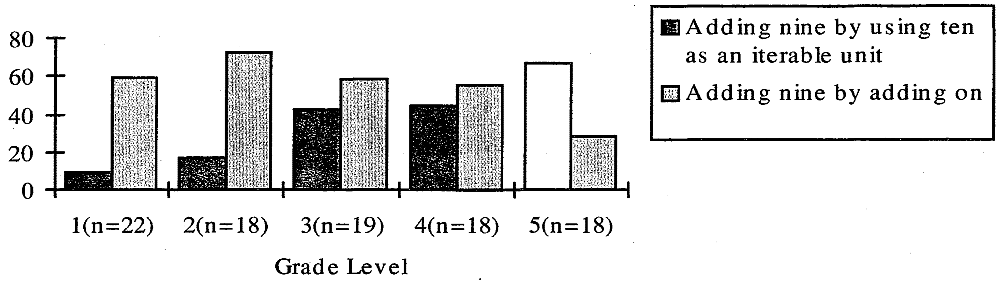


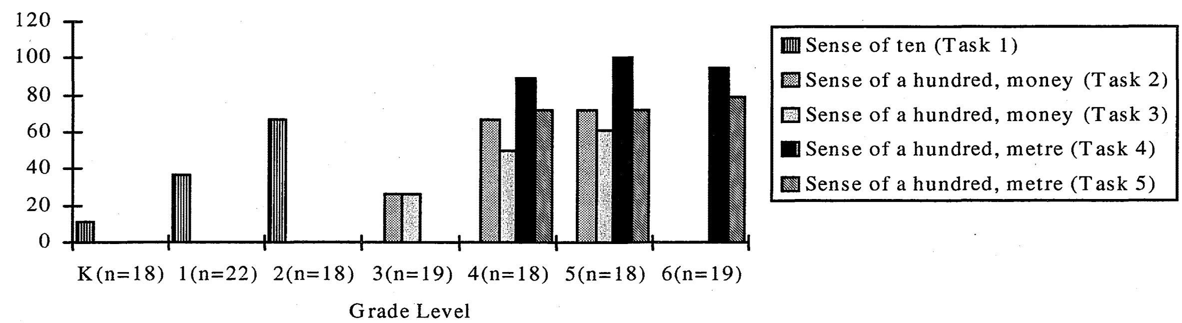
Figure 3 shows 33% of Grade 5 children still using unitary counting for Regrouping Task 2.

Figure 3 Regrouping Task 2: Addition with Concrete Material, add 9 to a Representation of 52 using bags of 10 shells and single shells. Solution strategies for Regrouping Task 2, Addition: Percentage of sample giving correct responses, by strategy use.



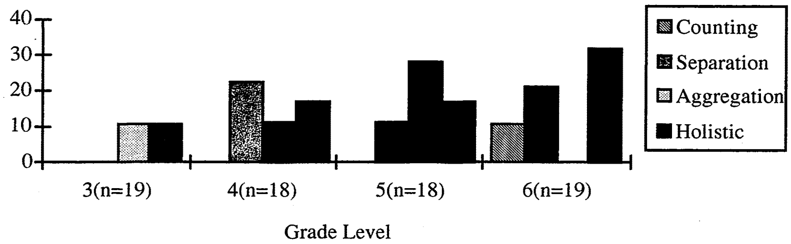
The results of the number sense tasks compared in Figure 4 also showed that children in Grades 2 and 3 exhibited low performance on using the part-whole relationships with ten and one hundred.

Figure 4 Number Sense Tasks 1 to 5: Addition to Ten and Applications Performance on Number Sense Tasks 1 to 5: Percentage of Sample Partitioning Tens and Hundreds, on Separate Tasks. Number Sense Tasks 1 to 5: Addition to ten and applications



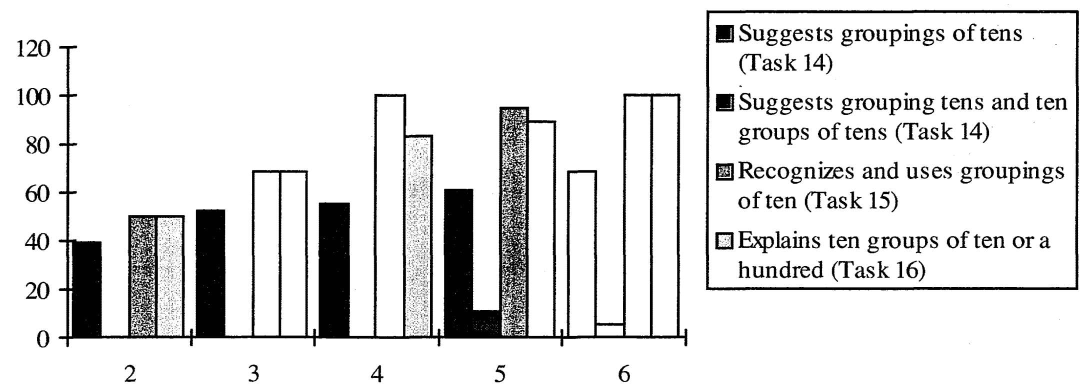
A child who uses ten as a singleton unit might be able to recite the decade numbers but makes no sense of the increments of ten - the units of one and ten co-exist but are not coordinated. The results of this study (Figure 5) show that 22% of Grade 5 children could not deal with two different units simultaneously. Approximately a third of Grade 6 children could not successfully add two 2-digit numbers mentally, where the first number was represented with pregrouped material (Regrouping Task 7).

Figure 5 *Regrouping Task 7: Addition involving ones, tens and hundreds (245 + 98), the first addend shown as pregrouped material. Solution strategies for Regrouping Task 7, Addition: Percentage of sample giving correct responses, by strategy use.*



Another third of the Grade 6 children used counting or separation strategies. The existence of a significant number of children using counting and separation strategies could be explained by their strategies reflecting classroom instruction that commonly emphasises unitary counting in the early years and written procedures for algorithms in the later grades. There was a substantial number of students in Grades 2 and 3 (50% and 32% respectively) who were not successful in recognising and using groupings of ten to quantify a collection of objects shown in Figure 6.

Figure 6
Structure Tasks 14 to 16: Find the Number of Marks in a picture(144 marks randomly drawn).Performance on Structure Tasks 14 to 16, Groupings: Percentage of sample suggesting and recognising someone else's groupings of ten as a grouping number and as a grouping of groupings number.



Item response analysis also demonstrated the lack of progress made by children on many tasks over Grades 4 to 6. The difficulties that children experience with understanding the structure of the number system was further highlighted by the decline in performance shown by Grade 6 children when counting using groups of 10 x 10, repeated use of groupings of 10, suggesting the use of 10 groups of ten, and interpreting zero as a place holder.

DISCUSSION OF RESULTS

It appears that many children can only recognise numbers in terms of additive properties rather than a combination of additive and multiplicative properties. Many children in this study did not realise that multiunits are related through multiplication or that they can be exchanged. In a recent study, Clark and Kamii (1996) reported that although some children develop multiplicative thinking as early as Grade 2, most children still could not demonstrate consistent multiplicative thinking in Grade 5. The present study confirms these results. A substantial minority (about 20%) of the Grade 3 students had developed such an intuitive understanding of powers of ten that they could use the recursive multiplicative structure of the array of 10,000 dots to count the number of dots successfully. By Grade 6, however, there was still a significant number who could not quantify 10 groups of 10 groups of 10 (10x10x10).

Children appeared to have had little experience recognising or using arrays. There was over a third of Grade 6 children who could not use their recognition of the pattern of hundreds in an array of dots to quantify the whole collection. Many children did not know how to use recognised groups of one hundred objects to quantify a collection of 10 000 objects. They did not make the connection that there was a need for equal grouping or multiplication in order to quantify the collection.

The idea that the numeration system is additive in the simplest way (multi-digits represent the total of the face value of the digits) is very strong among Grade 1 children (95.5%) and persists with some children until Grade 4 (22%) and probably beyond. A surprisingly large number of children (61% in Grade 2 through to 31% in Grade 6) was not able to suggest grouping by tens as a means of quantifying a collection. Many children seem slow to grasp these basic elements of grouping tens, formation of multiunits and the way the position of digits plays a role in terms of quantity.

CONCLUSIONS

This study showed that understanding of numeration developed slowly over the Kindergarten to Grade 6 period and that very few children were able to generalise the multiplicative structure of the system. There was evidence of the use of abstract counting strategies by some Kindergarten children, but the performance on counting tasks in the upper grades was still poor for many children. Although the performance on the estimation task was consistently good across the grades, there was lower than expected performance on many number sense tasks because of a high reliance on counting strategies rather than holistic strategies. Although there was good performance on using grouping in quantifying and building grouped material, there were indications that children did not understand the significance of ten in the number system. This understanding is critical to their further development of understanding and use of the numeration system.

The study highlights the difficulties that primary school children have in understanding the complex nature of the number system. Children did not understand the multiplicative relationships within the system that are the basis of place value structure and the patterns in the counting sequence. Children could count and group in tens but did not relate these processes to a base ten structure.

Implications

Many children were shown not to have developed a basic knowledge of the numeration system outside the ones to thousands range by the end of Grade 6. As Dienes' blocks are used widely in New South Wales classrooms, it seems that there might be a connection between children's limited number knowledge and their experiences with the standard representations and interpretations of the blocks. Boulton-Lewis & Halford (1992) argued that concrete materials are only useful if "children clearly recognize the correspondence between the structure of the material and the structure of the concept" (p. 21). Although it was not a specific focus of this study, it appears that children who are familiar with Dienes' blocks may have a limited understanding of the structure inherent in the blocks.

There is an emphasis in schools on multiplication tables and algorithms, but children are not sufficiently exposed to the idea that the base ten system is based on multiplication by 10. Multiplication and division need to be more closely linked, and more experiences bringing out the recursive nature of repeated groupings needs to be provided.

Attempts to develop professional support for teaching and learning to improve the understanding of numeration are currently in progress both in Australia and overseas. The NSW Department of Education and Training has two school-based professional development initiatives focused on numeracy. Both the *Count Me In Too* (CMIT) Project for K-3 grade teachers and the *Counting On* Project for Year 7/8 Secondary teachers (for less able students) aim to facilitate better understanding of students' mathematical strategies when using the number system.

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